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Electromagnetic Energy-Momentum Tensor and Elementary Classical Point Charges

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An energy-momentum tensor of electromagnetic fields associated with elementary classical point charges is constructed. This tensor represents finite amounts of energy-momentum and of corresponding fluxes. The differences between this tensor and the ordinary one, which is associated with continuously distributed charged matter, stem from the elementary nature of point charges. The new tensor is free of the 4/3 problem of the momentum of a point charge. Implications of the third-order Lorentz–Dirac equation are discussed.

1. INTRODUCTION

Classical electrodynamics of continuously distributed charged matter is a self-consistent theory. In this theory one can apply methods of differential calculus and derive expressions which are linear in an infinitesimal charge element dq. This procedure justifies ignoring all terms depending on higher powers of dq. Constituents of this version of classical electrodynamics are fields and continuous matter or matter particles whose charge is distributed continuously inside their volume.

The discovery of the electron provided a reason for the introduction of elementary point charges into classical electrodynamics. Theoretical developments support this attempt. Following the relativistic arguments of Landau and Lifshitz (1975) as well as the quantum mechanical ones of Rohrlich (1965), one is motivated to introduce elementary classical point charges into the theory. The elementary nature of a point charge means that an interaction between its charged constituents is unphysical. Several kinds of problems follow the incorporation of these entities into classical electrodynamics.

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Some problems are associated with their law of motion, namely, the third-order Lorentz–Dirac (LD) equation and the infinities of its runaway solutions (see, e.g., Dirac, 1938; Landau and Lifshitz, 1975; Rohrlich, 1965; and Pearle, 1982). Other problems emerge from the infinitely strong field at the neighborhood of a point charge.

Recently, Comay (1990*a*,*b*) analyzed scattering processes of classical point charges and used mathematical arguments in a proof showing that runaway solutions do not pertain to the LD equation. Moreover, alternative second-order equations of motion of elementary point charges have been proved to be unphysical (Huschilt and Baylis, 1974; Comay, 1987*a*, 1990*c*,*d*). These results support the acceptance of the LD equation as the law of motion of elementary classical point charges and encourage the hope that classical electrodynamics of elementary point charges can be put on a self-consistent basis. The main objective of the present work is to show that problems associated with the infinitely strong field at the vicinity of an elementary classical point charge can be settled. The analysis assumes that the LD equation is the law of motion of elementary classical point charges.

In the following discussion a classical particle whose charge is distributed continuously inside its volume is called a C-charge. Charge elements of C-charges discussed here are held firmly at their relative place by mechanical forces, and particles can, for practical purposes, be considered rigid bodies. Elementary classical point charges are called P-charges. The principle stating that there is no meaning to interactions of a constituent of a P-charge with other constituents of the same P-charge is called the elementariness principle. This principle plays a crucial role in the discussion carried out here.

Expressions are written in units where the speed of light takes the value $c = 1$. Greek indices range from 0 to 3 and Latin ones run from 1 to 3. The metric $g_{\mu\nu}$ is diagonal and its entries are $(1, -1, -1, -1) \cdot \gamma = (1 - v^2)^{-1/2}$. Here $F^{\mu\nu}$ is the antisymmetric tensor of electromagnetic fields. The symbol $_{,\nu}$ denotes the partial differentiation with respect to x^ν and an upper dot denotes the differentiation with respect to the particle's invariant time τ. The terms v^μ and a^μ designate 4-velocity and 4-acceleration of a particle, respectively.

The work is organized as follows. Section 2 discusses energy-momentum tensors of systems of C-charges. Problems that follow the introduction of P-charges are pointed out in Section 3. Section 4 analyzes electromagnetic fields as sums of appropriate constituents. Section 5 discusses a system of P-charges for which a field energy-momentum tensor is constructed. This tensor represents finite amounts of energy-momentum and of corresponding fluxes and its associated 3-momentum is free of the 4/3 problem. Topics pertaining to the compatibility of the results are discussed in Section 6. Concluding remarks are the contents of the last section.

2. ENERGY-MOMENTUM TENSORS

The symmetric energy-momentum tensor of an electromagnetic field associated with continuously distributed charged matter is a well-established quantity (Landau and Lifshitz, 1975)

$$T^{\mu\nu}_{(f)} = \frac{1}{4\pi}\,(F^{\mu\alpha}F^{\beta\nu}g_{\alpha\beta} + \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}) \tag{1}$$

The entries of (1) represent energy-momentum density and current, respectively. [For example, $T^{00} = (E^2 + B^2)/8\pi$ is the field's energy density.] In the case of continuously distributed charged matter, electromagnetic fields take regular values, entailing regular expressions for energy-momentum density and flux. Values of this kind make it possible to construct a self-consistent theory. In particular, Landau and Lifshitz (1975) show that one can write a Lagrangian of the system, derive Maxwell equations

$$F^{\mu\nu}{}_{,\nu} = -4\pi J^{\mu} \tag{2}$$

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0 \tag{3}$$

the Lorentz force

$$ma^{\mu} = qF^{\mu\nu}v_{\nu} = \int_V F^{\mu\nu}J_{\nu}\,d^3r \tag{4}$$

and the following relation:

$$(T^{\mu\nu}_{(f)} + T^{\mu\nu}_{(m)})_{,\nu} = 0 \tag{5}$$

Here V is the volume of the charge whose acceleration is derived from the Lorentz force and the quantities in the intermediate expression of (4) are the appropriate mean values obtained from the integration over V. The term $T^{\alpha\beta}_{(m)} = \mu v^{\alpha}v^{\beta}/\gamma$ is the matter energy-momentum tensor and μ denotes the matter density. Relation (5) proves local conservation of energy-momentum. It shows that, at charge-free volume elements, the field energy-momentum is conserved, whereas at volume elements where charge density is nonzero, the sum of field and matter energy-momentum tensors is a conserved quantity. This discussion shows that classical electrodynamics of continuously distributed charged matter can be considered a closed regular theory, so far as energy-momentum conservation is concerned.

3. PROBLEMS EMERGING FROM THE INTRODUCTION OF P-CHARGES

P-charges are not the same entities as C-charges. In the case of the latter kind of particles, every infinitesimal charge element interacts with all other charges, yielding the Lorentz force (4) which is linear in dq. On the other hand, the elementariness principle states that an interaction of a constituent of a P-charge with other parts of the same particle is inconsistent with the elementary nature of the particle. Hence, the satisfactory results obtained for the energy-momentum tensor of C-charges are not directly applicable to systems which consist of P-charges. This conclusion pertains to the fact that in the case of P-charges, one can consider neither finite charge densities nor an *infinitesimal* amount of charge interacting with *all* other charge quantities. The following problems emerge from the introduction of these new entities and are discussed in the present work:

1. The elementariness principle excludes the interactions of one part of a P-charge with the rest of this particle and may affect energy-momentum balance.
2. The strength of the electromagnetic field of a P-charge increases beyond all bounds as the distance between the particle and a point where the field is measured approaches zero.
3. Continuously distributed charged matter satisfies the Lorentz force (4), whereas a P-charge obeys the LD equation

$$\tfrac{2}{3}q^2\dot{a}^\mu = ma^\mu - qF^{\mu\nu}_{(\text{ext})}v_\nu - \tfrac{2}{3}q^2(a^\alpha a_\alpha)v^\mu \tag{6}$$

Unlike the second-order Lorentz force (4), which is written in terms of the *entire* field tensor, the LD equation is a third-order differential equation which depends on the particle's kinematic variables, on its own mass and charge, and on fields associated with *external* sources. The differences between the Lorentz force (4), which holds for C-charges, and the LD equation (6), which is the law of motion of P-charges, may induce changes in the definition of the field energy-momentum tensor.

Each of these points indicates that the derivation of energy-momentum conservation (5) may require modifications. In the rest of this work it is shown that a self-consistent energy-momentum tensor of electromagnetic fields of P-charges can be established. This tensor represents finite amounts of energy-momentum and of corresponding fluxes.

4. SPLIT OF ELECTROMAGNETIC FIELDS

The linearity of classical electrodynamics enables the split of fields into sums of appropriate terms. Remembering that the system is made of P-charges, one finds that the overall field tensor can be written as a sum of tensors, each of which is related to a distinct charge q_i. The Lienard–Wiechert expression for the fields of q_i is

$$\mathbf{E}=q_i\left[\frac{1-v^2}{(R-\mathbf{R}\cdot\mathbf{v})^3}(\mathbf{R}-R\mathbf{v})+\frac{1}{(R-\mathbf{R}\cdot\mathbf{v})^3}\mathbf{R}\times\langle(\mathbf{R}-R\mathbf{v})\times\mathbf{a}\rangle\right] \tag{7}$$

$$\mathbf{B}=\mathbf{R}\times\mathbf{E}/R \tag{8}$$

Here $\mathbf{R}$ denotes the radius vector from the retarded position of q_i to the point where the fields are calculated and $\mathbf{v}$ and $\mathbf{a}$ are the retarded velocity and acceleration of q_i, respectively. For obvious reasons, the first terms of (7) and of (8) are called velocity fields and the second ones are called acceleration fields. Henceforth, these fields are denoted by the symbols $F^{\mu\nu}_{(i,v)}$ and $F^{\mu\nu}_{(i,a)}$, where the subscript i is the index of the charge with which this field tensor is associated and v and a denote velocity fields and acceleration fields, respectively. In cases where the subscripts v and a are omitted, the tensor represents the overall field associated with q_i. If all subscripts are suppressed, then the tensor $F^{\mu\nu}$ is the tensor of the entire field. Using this notation, one can decompose the field tensor as follows:

$$F^{\mu\nu}\equiv\sum_i F^{\mu\nu}_{(i)}\equiv\sum_i F^{\mu\nu}_{(i,v)}+\sum_i F^{\mu\nu}_{(i,a)} \tag{9}$$

The following property of fields is used in the analysis carried out in the present work. It is well known that a physically meaningful quantity must, directly or indirectly, be related to measurement. Classical electromagnetic fields are not directly measurable entities, namely, these fields cannot be measured by a device made of electromagnetic fields alone. On the other hand, these fields can be measured by classical charges which accelerate by virtue of their interaction with fields. It follows that classical electromagnetic fields become meaningful only in circumstances where their interactions with charges may take place.

Measurability and the Maxwell equation (2) indicate the double role of charges with respect to fields. In the Maxwell equations a charge acts as a *source* of fields, whereas its equation of motion is used for *measuring* them. These two distinct roles of charges yield no problem if fields associated with one charge are measured by another charge. On the other hand, it is not clear what should be done in the case where one and the same charge takes the two roles. In other words, the problem is how to account for the interaction of a charge with itself.

In order to examine this problem, let us consider a C-charge and follow the analysis of Landau and Lifshitz (1975) of interactions of charges with fields. Later, the results found for the motion of C-charges are used for P-charges. The motion of a P-charge is characterized if each point on its trajectory is given as a function of time. An analogy between C-charges and P-charges can take place if a P-charge moves like the center of mass of a very small C-charge. The spatial size of C-charges used here is much smaller than the minimal distance between any two particles. Therefore, fields of external charges are practically uniform over the volume of each C-charge. Moreover, C-charges are considered stable and they are tested under conditions where effects of internal motion of their constituents (such as rotations or vibrations) can be ignored. Therefore, except at points that are very close to C-charges, fields take, for practical purposes, the same values as in corresponding systems of P-charges. Only this kind of system of C-charges is discussed below.

In the following lines it is shown how the analysis of Landau and Lifshitz (1975) yields the motion of the center of mass of C-charges. Consider a specific charge q_i and the following decomposition of fields:

$$F^{\mu\nu} = F^{\mu\nu}_{(\text{ext})} + F^{\mu\nu}_{(i)} \tag{10}$$

where

$$F^{\mu\nu}_{(\text{ext})} \equiv \sum_{j \neq i} F^{\mu\nu}_{(j)} \tag{11}$$

is the field tensor of all other charges except q_i.

In the analysis of the effect of (10) on the motion of the center of mass of q_i, one starts from the Lorentz force (4)

$$ma^{\mu} = \int_{V_i} F^{\mu\nu}_{(\text{ext})} J_\nu \, d^3r + \int_{V_i} F^{\mu\nu}_{(i)} J_\nu \, d^3r = q_i F^{\mu\nu}_{(\text{ext})} v_\nu + \int_{V_i} F^{\mu\nu}_{(i)} J_\nu \, d^3r \tag{12}$$

where the integration is carried out over the volume V_i of q_i. As stated above, $F^{\mu\nu}_{(\text{ext})}$ can be considered uniform over V_i and the first integral is straightforward. In the following analysis $F^{\mu\nu}_{(i)}$ is substituted by appropriate quantities, whereas $F^{\mu\nu}_{(\text{ext})}$ is left unchanged. For this purpose, the speed of light c is used explicitly. Expanding potentials in power series of c^{-1}, it is proved that if field quantities related to powers of c^{-3} and higher powers of c^{-1} are ignored, then one obtains the Darwin Lagrangian, which is written in terms of particles alone,

$$\begin{aligned} L = {} & \tfrac{1}{2} \sum_j m_j v_j^2 + \frac{1}{8c^2} \sum_j m_j v_j^4 - \sum_{j>l} \frac{q_j q_l}{R_{jl}} \\ & + \sum_{j>l} \frac{q_j q_l}{2c^2 R_{jl}} \left[\mathbf{v}_j \cdot \mathbf{v}_l + \frac{(\mathbf{v}_j \cdot \mathbf{R}_{jl})(\mathbf{v}_l \cdot \mathbf{R}_{jl})}{R_{jl}^2} \right] \end{aligned} \tag{13}$$

This expression depends on instantaneous radius vectors $\mathbf{R}_{jl}$ between coordinates of every pair of charge elements. This Lagrangian is invariant under translation and is independent of the fields. Applying it only to charge elements located within V_i, one finds that forces derived from (13) do not affect the motion of the particle's center of mass.

The incorporation of c^{-3} terms in the expansion shows that $F^{\mu\nu}_{(i)}$ yields the q^2-dependent terms of the LD equation (6). The analysis employs the Taylor expansion of $\rho(\mathbf{r}, t-R/c)$ and of $\mathbf{J}(r, t-R/c)$ in power series of R/c. Therefore, one finds that higher powers of the expansion can be ignored because the discussion is restricted to charges located within a C-charge, whose spatial dimensions are very small. Indeed, in this case, R is not greater than the particle's size, and terms containing c^{-k}, where $k>3$, are multiplied by R^n, where n is a positive integer.

The foregoing discussion proves that the interaction of a very small C-charge with its own fields $F^{\mu\nu}_{(i)}$ can be replaced by the q^2-dependent terms of the LD equation (6). Hence, if one uses the LD equation (6), then the interactions of this C-charge with its own fields are already taken into account by means of the terms proportional to q^2. This is why the LD equation (6) (which is derived from the Lorentz force) is written in terms of *external* fields $F^{\mu\nu}_{(\mathrm{ext})}$, unlike the Lorentz force (4), which depends on the entire field tensor $F^{\mu\nu}$. The correspondence between the motion of P-charges and that of C-charges whose spatial size is very small is used in the following sections.

5. SPLIT OF THE ELECTROMAGNETIC ENERGY-MOMENTUM TENSOR

Let us examine the energy-momentum tensor of fields of C-charges. Later, by means of the elementariness principle, a corresponding tensor of fields of P-charges is derived. The field energy-momentum tensor (1) is a quadratic function of $F^{\mu\nu}$. It can be split into a sum of terms, each of which depends on particular field quantities. To this end, $T^{\mu\nu}$ is written as a function of two variables. Thus, for example,

$$T^{\mu\nu}(F^{\pi\rho}_{(1)}, F^{\sigma\tau}_{(2)}) \equiv \frac{1}{4\pi}\,(F^{\mu\alpha}_{(1)}F^{\beta\nu}_{(2)}g_{\alpha\beta} + \tfrac{1}{4}F^{\alpha\beta}_{(1)}F_{(2)\alpha\beta}\,g^{\mu\nu}) \tag{14}$$

Using the notation of (9) and (14), one can cast (1) into the following sum:

$$\begin{aligned} T^{\mu\nu} = {} & \sum_{i\neq j} T^{\mu\nu}(F^{\pi\rho}_{(i)}, F^{\sigma\tau}_{(j)}) + \sum_i T^{\mu\nu}(F^{\pi\rho}_{(i,a)}, F^{\sigma\tau}_{(i,a)}) \\ & + \sum_i T^{\mu\nu}(F^{\pi\rho}_{(i,v)}, F^{\sigma\tau}_{(i,a)}) + \sum_i T^{\mu\nu}(F^{\pi\rho}_{(i,a)}, F^{\sigma\tau}_{(i,v)}) \\ & + \sum_i T^{\mu\nu}(F^{\pi\rho}_{(i,v)}, F^{\sigma\tau}_{(i,v)}) \end{aligned} \tag{15}$$

An important property of the terms of (15) is their relation to incoming and outgoing fluxes of energy-momentum with respect to a small volume enclosing a C-charge q_i. Consider a small sphere S having a radius r and whose center is at a retarded position of q_i. The examination is carried out in an inertial frame where the C-charge is temporarily at rest at the retarded position. The radius r is much smaller than the distance between q_i and any other charge q_j. Hence, $F^{\mu\nu}_{(j)}$ can be considered uniform inside S and the dominant part of the first term on the right-hand side of (15) behaves there like r^{-2}. Similarly, since $F^{\pi\rho}_{(i,a)}$ behaves like r^{-1}, one finds that the second term of (15) behaves like r^{-2}, too. On the other hand, the other terms on the right-hand side of (15) behave like r^{-3} and r^{-4}, respectively.

Except at the volume of q_i, the sphere S is matter-free. Therefore, the field part of energy-momentum is conserved at points inside S which are outside q_i. Hence, terms of (15) that behave like r^{-2} represent outgoing or incoming fluxes of energy-momentum. On the other hand, terms decaying like r^{-3} or r^{-4} cannot represent such fluxes. It follows that the last three terms on the right-hand side of (15) must be associated with fluxes of energy-momentum exchanged between charge constituents of q_i (see, e.g., Teitelboim *et al.*, 1980). This discussion shows the physical meaning of each term of (15) in cases where it is associated with a system made of C-charges.

Let us examine an analogous system where C-charges are replaced by P-charges. The charge and mass of each P-charge equal the corresponding quantities of the C-charge replaced by it. The same is true for the respective kinematic variables of the particles. These relations guarantee that mass and charge as well as initial conditions are the same in the two systems.

Due to the elementariness principle, it is meaningless to consider an exchange of energy-momentum between constituents of a P-charge. The introduction of the LD equation (6) as the law of motion of P-charges is consistent with this principle because this equation does not depend on self-fields. Moreover, as shown in Section 4, the LD equation is a formula derived from the Lorentz force. This formula holds for systems where the spatial size of each C-charge is small enough. Therefore, due to the fit of self-mass, charge, and initial conditions, one finds that the motion of the center of mass of C-charges of the first system is the same as that of the corresponding P-charges of the second system. This outcome means that, in the LD equation (6), self-interaction of P-charges is eliminated, whereas the outer world "sees" P-charges move like corresponding C-charges.

An analogous step, where effects associated with self-interaction of P-charges are eliminated, should be taken in the case where the form of the corresponding field energy-momentum tensor is discussed. Indeed, due to the elementariness principle, a constituent of a P-charge cannot exchange energy-momentum with other parts of the same P-charge. Hence, portions of the energy-momentum tensor representing this exchange, namely, the last three terms of (15), are physically meaningless. Therefore, in the case of P-charges, the field energy-momentum tensor is

$$T^{\mu\nu} = \sum_{i\neq j} T^{\mu\nu}(F^{\pi r}_{(i)}, F^{\sigma\tau}_{(j)}) + \sum_{i} T^{\mu\nu}(F^{\pi\rho}_{(i,a)}, F^{\sigma\tau}_{(i,a)}) \tag{16}$$

It can easily be seen that the two-particle interaction part of (1) agrees with that of (16). The same is true for the intensity of the radiation part of these tensors. Moreover, (16) behaves like r^{-2} near every P-charge. It represents finite amounts of energy-momentum fluxes exchanged between fields and P-charges and its energy-momentum density yields finite quantities. Therefore, it requires no further mathematical manipulations called renormalization or cutoff procedures. The derivation of (16) is based on general principles, namely the elementariness principle and the measurability requirement. These principles state that one cannot measure the force exerted by a constituent of an elementary particle on other constituents of the same particle. Hence, there is no physical meaning to energy-momentum *flux* between such constituents, namely, to the current components $T^{\mu i}$ of the last three terms of (15). Covariance requirements mean that the *entire* tensor associated with this flux should be deleted. This operation entails the removal of the infinities associated with energy-momentum *densities* $T^{\mu 0}$ of the same tensor.

The results of this section settle also another old problem of elementary classical point charges. This problem is known as the 4/3 factor of momentum obtained from a Lorentz transformation of fields of an elementary classical charge. Several articles discussing this topic have been published recently (Boyer, 1982, 1985; Rohrlich, 1982; Bialynicki-Birula, 1983; Campos and Jimenez, 1986; Comay, 1991). The problem emerges from a comparison of the transformed 4-momentum of the system with the momentum of the transformed fields. Considering the nonrelativistic limit of a Lorentz transformation of the field 4-momentum of a spherical charge, one finds that the corresponding 3-momentum is 4/3 times the required quantity. Bialynicki-Birula (1983) shows by means of an example that an incorporation of the Poincaré stress settles the problem. However, this approach is inapplicable to an elementary classical point charge because the elementariness principle is inconsistent with an additional internal mechanical stress.

The energy-momentum tensor (16) is free of the 4/3 contradiction. In the case of a single inertially moving P-charge, the electromagnetic 4-momentum vanishes and its entire 4-momentum is ascribed to its mechanical part, namely $p^{\mu} = m\gamma(1, v_x, v_y, v_z)$. Obviously, an application of a Lorentz transformation to such a particle yields results that are consistent with relativistic requirements.

Using field decomposition, one can view (16) as follows. The first term of (16) is the interaction part of the field energy-momentum tensor (1)

$$T^{\mu\nu}_{(I)} = \sum_{i \neq j} T^{\mu\nu}(F^{\pi\rho}_{(i)}, F^{\sigma\tau}_{(j)}) \tag{17}$$

The radiation part of (17) together with the last term of (16) give the energy-momentum tensor of radiation fields

$$T^{\mu\nu}_{(\mathrm{rad})} = \sum_{i,j} T^{\mu\nu}(F^{\pi\rho}_{(i,a)}, F^{\sigma\tau}_{(j,a)}) \tag{18}$$

It follows that the interaction part of the radiation fields is contained in (17) as well as in (18).

6. THE PROBLEM OF COMPATIBILITY OF RESULTS

The problem of correspondence between the theory of P-charges and the well-known local energy-momentum conservation (5) of C-charges deserves attention. This issue is examined first at charge-free points of space-time. Later the problem of interaction of P-charges with fields is addressed.

A comparison of the energy-momentum tensor (16) with (15) shows that (16) contains all terms of (15) except the last three. Following Teitelboim *et al.* (1980), one finds that the deleted terms behave like r^{-3} and r^{-4}, respectively and the 4-divergence of their sum vanishes outside the charge's world line. The same is true for the second term of (15). Since the 4-divergence of the entire field tensor (15) vanishes there, one concludes that the 4-divergence of (16) vanishes at all charge-free points of space-time. This conclusion shows that (16) is consistent with energy-momentum conservation at charge-free space-time points.

Let us consider points of space-time on the world line of a P-charge q_i. Here $\mathbf{r}_i$ denotes the position of q_i at $t=0$, and S is a small sphere centered at $\mathbf{r}_i$. The sphere S contains no charge except q_i. The 4-divergence of the interaction term (17) on the right-hand side of (16) is discussed first. Evidently, if a term of the summation does not depend on fields of q_i, then, as shown above, the corresponding 4-divergence vanishes. Therefore, the required 4-divergence is

$$T^{\mu\nu}_{(I),\nu} = T^{\mu\nu}(F^{\pi\rho}_{(i)}, F^{\sigma\tau}_{(\mathrm{ext})})_{,\nu} + T^{\mu\nu}(F^{\pi\rho}_{(\mathrm{ext})}, F^{\sigma\tau}_{(i)})_{,\nu} \tag{19}$$

where (11) is substituted for the summation on j ($j \neq i$).

At points inside S, the Maxwell equation (2) yields for the fields used in (19)

$$F^{\mu\nu}_{(\mathrm{ext}),\nu} = 0 \tag{20}$$

$$F^{\mu\nu}_{(i),\nu} = -4\pi q_i v^\mu \delta(\mathbf{r} - \mathbf{r}_i) \tag{21}$$

Using these expressions together with the homogeneous Maxwell equation (3), one follows Landau and Lifshitz's (1975) calculation of the 4-divergence of the complete energy-momentum tensor of fields and obtains

$$T^{\mu\nu}_{(I),\nu} = -q_i F^{\mu\nu}_{(\text{ext})} v_\nu \delta(\mathbf{r} - \mathbf{r}_i) \tag{22}$$

This result balances the interaction of a P-charge with *external* fields, as used in the LD equation (6).

The 4-divergence of the last term of (16) at the position of q_i is (Teitelboim *et al.*, 1980)

$$T^{\mu\nu}(F^{\pi\rho}_{(i,a)}, F^{\sigma\tau}_{(i,a)})_{,\nu} = -\tfrac{2}{3} q_i^2 (a^\alpha a_\alpha) v^\mu \tag{23}$$

This quantity compensates the last term of the LD equation (6).

The foregoing results show that the LD equation (6) introduces an additional term, $\frac{2}{3}q^2\dot{a}^\mu$, which is inconsistent with local energy-momentum balance of the system (provided the definition of kinetic 4-momentum is not altered). The 0-component of this quantity is sometimes called the "Schott energy" of the "induction field energy" (Coleman, 1982). Coleman (1982) shows that the existence of this quantity is not incompatible with physical requirements in scattering processes where q_i begins and ends with inertial motion as well as in cases of a periodic motion. These properties of the LD equation explain the successful tests carried out in the cases of the circular uniform motion of charges (Comay, 1987*b*) and in a scattering process (Huschilt and Baylis, 1976) where the asymptotic inertial motion is proved (Comay, 1990*a*,*b*).

Relying on these results, one can follow Coleman (1982) as well as Teitelboim *et al.* (1980) and arrive at a local conservation law by means of a redefinition of the particle's mechanical 4-momentum. Writing

$$\bar{p}^\mu \equiv m v^\mu - \tfrac{2}{3} q^2 a^\mu \tag{24}$$

one finds from (6), (22), and (23) that $\bar{p}^\mu$ can be considered part of a locally conserved formalism.

However, the definition (24) of mechanical 4-momentum stems from an *ad hoc* approach and its basis is not as solid as that of ordinary mechanical 4-momentum. In particular, it is interesting to note that the unbalanced term of the LD equation (6), namely $\frac{2}{3}q^2\dot{a}^\mu$ [with which the last term of (24) is associated], cannot be derived from a Lagrangian. In order to yield a third-order equation of motion of charged particles, their Lagrangian must take the form

$$L = L(\dot{x}^\mu, \ddot{x}^\mu, A^\mu_{(\text{ext})}) \tag{25}$$

where $\dot{x}^\mu = v^\mu$ and $\ddot{x}^\mu = a^\mu$. Here x^μ is omitted from the Lagrangian's variables because an explicit dependent on x^μ is inconsistent with space-time homogeneity. As shown by Jackson (1975), the analysis becomes simpler if a manifestly covariant treatment is carried out and the particle's invariant time τ is used as the independent variable. Let us take, for example, the term ma^μ of (6) as an illustration of this approach. This term can be obtained from a Lagrangian containing a function of the following Lorentz scalar $v^\alpha v_\alpha$. Here all four components of v^μ can be considered independent variables and the constraints $v^\mu v_\mu = 1$ is introduced by means of a Lagrangian multiplier (Barut, 1965). This example shows that the application of τ as the independent variable means that the Lagrangian can be written in a form where all terms are Lorentz scalars, a property which guarantees that the equations of motion take the same form in all inertial frames.

In (25), the contribution of the potential to the equation of motion is well known and is omitted here for the sake of brevity. The free-particle part of the equation of motion obtained from the Lagrangian (25) is given by Akhiezer (1962),

$$-\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} + \frac{d^2}{d\tau^2}\frac{\partial L}{\partial \ddot{x}^\mu} = 0 \tag{26}$$

This Lagrangian depends on two scalar functions:

$$\zeta = \tfrac{1}{2} v^\alpha v_\alpha, \qquad \eta = v^\alpha a_\alpha \tag{27}$$

(Note that a variable of the form $a^\alpha a_\alpha$ yields a fourth-order equation.) Using ζ, η, and a Lagrangian multiplier λ, one puts the free-particle part of the Lagrangian in the following form:

$$L(\zeta, \eta) + \lambda (v^\alpha v_\alpha - 1) \tag{28}$$

The constraint $2\zeta = v^\alpha v_\alpha = 1$ yields

$$\dot{\zeta} = \eta = \frac{d}{d\tau}\frac{1}{2} = 0 \tag{29}$$

This relation and its derivatives are used in the following calculation.

Evidently, a variation of λ yields (29). Substituting (28) into (26), one finds

$$\begin{aligned}
&-\frac{d}{d\tau}\frac{\partial L(\zeta,\eta)}{\partial v^\mu}+\frac{d^2}{d\tau^2}\frac{\partial L(\zeta,\eta)}{\partial a^\mu}-\frac{d}{d\tau}(2\lambda v_\mu)\\
&=-a_\mu L'_\zeta-v_\mu\eta L''_{\zeta\zeta}-v_\mu\dot\eta L''_{\zeta\eta}-\dot a_\mu L'_\eta-a_\mu\eta L''_{\eta\zeta}-a_\mu\dot\eta L''_{\eta\eta}\\
&\quad+\dot a_\mu L'_\eta+2a_\mu\eta L''_{\eta\zeta}+2a_\mu\dot\eta L''_{\eta\eta}+v_\mu\dot\eta L''_{\eta\zeta}+v_\mu\ddot\eta L''_{\eta\eta}\\
&\quad+v_\mu\eta^2L'''_{\eta\zeta\zeta}+2v_\mu\eta\dot\eta L'''_{\eta\eta\zeta}+v_\mu\dot\eta^2L'''_{\eta\eta\eta}-2\dot\lambda v_\mu-2\lambda a_\mu\\
&=-a_\mu L'_\zeta-2\dot\lambda v_\mu-2\lambda a_\mu
\end{aligned}\tag{30}$$

Here partial derivatives of L are denoted by L'_ζ, etc., and η is written in place of $\dot\zeta$. Two terms $\pm\dot a_\mu L'_\eta$ cancel each other. Many other terms of the intermediate expression vanish by virtue of (29) and its derivatives. The final result consists of three terms and is independent of $\dot a_\mu$ and of L'_η. It means that the scalar variable $\eta=v^\alpha a_\alpha$ does not affect the free-particle part of the equation of motion and can be eliminated from the Lagrangian (28). This outcome shows that the kinematic term $\frac{2}{3}q^2\dot a^\mu$ of (6) cannot be derived from a Lagrangian depending only on the particle's 4-velocity and its derivatives.

The discussion carried out in this section shows that local balance of energy-momentum is obtained from the energy-momentum tensor (16) at charge-free points of space-time. If one does not alter the definition of kinetic 4-momentum, then temporal nonconservation of energy-momentum emerges from the matter part of the theory where the third-order LD equation holds. A feature like this is already known in conditions which are outside the domain of validity of classical physics, namely where quantum mechanics, with its inherent uncertainties, describes nature. Obviously, local nonconservation of energy-momentum cannot be used as an argument for rejecting the LD equation, provided the *entire* process conserves energy-momentum. This property of the LD equation has been proved recently for scattering processes (Comay, 1990*a*,*b*).

7. CONCLUDING REMARKS

The main objective of the present work is the derivation of the field energy-momentum tensor of a system of elementary classical point charges that represents finite amounts of energy-momentum fluxes. The derivation relies on two principles: the elementariness of point charges and the dependence of measurability of fields on charges. These principles lead to the energy-momentum tensor (16), which represents finite amounts of energy-momentum *fluxes*. Similarly, an integration of the corresponding energy-momentum *densities* yields finite quantities because (16) behaves like r^{-2} near every point particle. The tensor (16) satisfies energy-momentum conservation at charge-free points of space-time. Moreover, it contains the entire interaction part (17) of (15) as well as all of its radiation terms (18).

The terms not included in the resultant tensor (16) represent energy-momentum currents exchanged between charge constituents of the same particle. This interaction is meaningful in the case of a particle whose charge is distributed continuously inside its volume. However, in the case of an elementary classical point charge, these quantities are *unmeasurable*, a property which shows that they are physically meaningless. This conclusion indicates that their removal from the final expression (16) should not cause a fundamental problem. These results are different from the ones obtained in other treatments, where the entire energy-momentum tensor (15) is kept and the ill effects of its unphysical parts are attempted to be removed by means of additional postulates.

Another aspect of this problem is the 4/3 factor of momentum of transformed fields. In the case of an elementary classical point charge, one finds that the energy-momentum tensor (16) is free of this problem.

The derivation of the energy-momentum tensor (16) is based upon general principles, namely the elementary nature of a classical point charge and the relation of physically meaningful quantities to measurements. The favorable results where infinite energy-momentum does not take place and the 4/3 problem is eliminated substantiate this approach.

It is also shown that if the form of the kinetic 4-momentum is not altered, then the theory of elementary classical point charges cannot reach the status of the theory of continuously distributed charged matter, where *local* conservation of energy-momentum (5) is proved. Moreover, the formal redefinition of the 4-momentum (24) cannot be obtained from a variational principle.

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